A Brief on Normal Numbers

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Primarily inspired by a discussion in Professor Steven Strogatz's Nonlinear Dynamics class, I discovered the concept of normal numbers.

Definition 1 (Simply normal)**.** A number $a \in \mathbb{R}$ is said to be *simply normal* in base $b \in \mathbb{N}$ if its infinite sequence expansion is distributed uniformly, i.e., for each digit $d \in \{0, ..., b-1\}$, the probability of it appearing as any given digit is $\frac{1}{b}$.

A similar definition also exists for being *normal in base b*, which intuitively is that any finite combination of digits (e.g., your phone number, your favorite number, DOB, etc.) only occurs for a finite length and is proportional to this rule. It is a stronger notion of normality than simply normal.

Definition 2 (Normal in base b)**.** A number $a \in \mathbb{R}$ is *normal in base b* if, for any $n \in \mathbb{N}$, all combinations of n digits has density $\frac{1}{b^n}$ in a.

This notion is similar to having each digit be i.i.d. uniform in the infinite expansion. In fact, the thought seems to have sprung on related notions of uniform randomness, such as in Borel (1909).

Numbers that are simply normal in all bases $b \geq 2$ are called *normal*. Almost all (save for a set of measure zero) real numbers are normal. In fact, Borel (1909) established what is now called the Borel-Cantelli Lemma which can be used to prove this to be true. Furthermore, following from the Borel-Cantelli Lemma, non-normal reals make up a set of measure zero. (Fun fact: as recounted in Khoshnevisan (2006), Borel's original proof of this fact had a large error that Faber and Hausdorff separately resolved a year after the original 1909 work was published).

Intriguingly, it is quite difficult to prove that a number is normal, even if we suspect it to be so. Examples include the usual suspects ($\sqrt{2}, \pi, e$). Furthermore, as is remarked throughout Khoshnevisan (2006), non-normal numbers can have just as complicated structures as normal ones. Sure, examples like 2 or $\frac{1}{17}$ are trivially non-normal numbers, but just because a number is non-normal does not mean it is not strange. An example I like that he brings up is from Cassels (1959).

Theorem 1 (Cassels' Theorem)**.** *Consider $f : [0,1] \to \mathbb{R}$ such that*

$$f(x) \equiv \sum_{i=1}^{\infty} \frac{x_i}{3^i}$$

where x_i denotes the i-th digit of x where x itself is written in base 2 (so $x_i \in \{0,1\}$). Then, for $x \in [0,1]$ a.e., $f(x)$ is simply normal to every base b that is not a power of 3.

An immediate remark is that clearly, it is not normal in base $3, 9, 27, ...$ (i.e., any base $b = 3^k$ for some $k \in \mathbb{N}$). Therefore, for almost all of $x \in [0,1]$, we have that $f(x)$ is non-normal.

References

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