Nonstandard Analysis Lecture

Andres Aradillas Fernandez

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Filters

Definition 1 (Filter)**.** Let I be a nonempty set. Let $\mathcal{P}(I)$ be the power set of I such that

$$\mathcal{P}(I) = \{A : A \subseteq I\}$$

A *filter* on I is a nonempty collection $\mathcal{F} \subseteq P(I)$ of subsets of I such that:

1. If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$
2. If $A \in \mathcal{F}$ and $A \subseteq B \subseteq I$, then $B \in \mathcal{F}$

Therefore, we can show $B \in \mathcal{F}$ by showing $A_1 \cap A_2 \cap ... \cap A_n \subseteq B$ when $A_i \in \mathcal{F}$

Definition 2 (Proper Filter)**.** A proper filter is a filter $\mathcal{F}$ such that $\emptyset \notin \mathcal{F}$.

Definition 3 (Ultrafilter)**.** An ultrafilter is a filter $\mathcal{F}$ where $\forall A \subseteq I$, either $A \in \mathcal{F}$ or $A^c \in \mathcal{F}$, but not both.

Examples of Filters:

1. $\mathcal{F}^i = \{A \subseteq I : i \in A\}$, called the principal ultrafilter.
2. $\mathcal{F}^{co} = \{A \subseteq I : I - A \text{ is finite}\}$, called the cofinite filter. NOT an ultrafilter (ex: neither the evens nor the odds are in the filter, but they are complements of each other). Cofinite filter on finite I is all of P(I)
3. $\mathcal{F}^{\mathcal{H}} = \{A \subseteq I : A \supseteq B_1 \cap ... \cap B_n \text{ for some } n \text{ and some } B_i \in \mathcal{H}\}$ where $\mathcal{H} \subseteq \mathcal{P}(I)$, called the filter generated by $\mathcal{H}$.

Construction of the Hyperreals

Corollary 1. Without proof, I assert that any infinite set has a nonprincipal ultra filter. This notion of an ultrafilter creates a notion of "largeness" that will become important.

Definition 4 (Notion of Equality/"Almost Everywhere")**.** We can create a notion of equality for sequences as follows. Fix a nonprincipal ultrafilter $\mathcal{F}$ on $\mathbb{N}$. Consider real sequences $r, s \in \mathbb{R}^{\mathbb{N}}$. We can define $\equiv$ as:

$$\{r_n\} \equiv \{s_n\} \text{ iff } \{n \in \mathbb{N} : r_n = s_n\} \in \mathcal{F}$$

We denote this as $r \equiv s$.

Definition 5 (Definition of ${}^\star\mathbb{R}$)**.** Consider the equivalence class of a sequence $r \in \mathbb{R}^{\mathbb{N}}$ under $\equiv$, and denote it as $[r]$. Therefore:

$$[r] = \{s \in \mathbb{R}^{\mathbb{N}} \, : \, r \equiv s\}$$

Then, we can define the quotient set as:

$${}^\star\mathbb{R} = \{[r] \, : \, r \in \mathbb{R}^{\mathbb{N}}\}$$

We can then also define operations such as addition and multiplication in an element-by-element way that makes sense and is well-defined under varying representations of a particular sequence within the equivalence class of $[r]$.

Transfer Principle

Without proof, I assert the following two statements to be true about working with nonstandard numbers:

Universal Transfer If a property holds for all real numbers, then it holds for all hyperreal numbers.

Examples

Archimedes principle:

$$(\forall x \in \mathbb{R})(\exists m \in \mathbb{N})(x < m)$$

This transfers to:

$$(\forall x \in^\star \mathbb{R})(\exists m \in^\star \mathbb{N})(x < m)$$

Existential Transfer If there exists a hyperreal number satisfying a certain property, then there exists a real number with this property.

Examples: Suppose we have an extended hypersequence $\star s \, : \, \star\mathbb{N} \to \star\mathbb{R}$ that never takes infinitely large values:

$$(\exists y \in \star\mathbb{R})(\forall n \in \star\mathbb{N})(| \star s(n)| < y)$$

Using existential principle:

$$(\exists y \in \mathbb{R})(\forall n \in \mathbb{N})(|s(n)| < y)$$

Extension of Reals

Consider $\omega = (1, 2, \ldots)$. It can be seen as a new number, since $\forall r \in \mathbb{R}$, $\star r = (r, r, r, \ldots)$ has the property above, and yet, ω does not. Therefore, it exists in $\star\mathbb{N} - \mathbb{N}$, and so this extension is meaningful and novel.

Application to Economics

Rational choice theory models of large economies model aggregate consumers based on individual consumers. Without detail, we want the positive aggregate consumer to have convex preferences. However, rational choice theory requires convex preferences of individuals in order for the aggregate to be convex.

On the other hand, if we use the existential principle twice, utilizing nonstandard methods, then we can soften this assumption. The assumption of convex preferences of individuals is quite a strong one, and so this is a powerful result. For more, see Anderson (2008).