Reformulating the Hyperbolic Angle for Hyperbolic Trigonometric Functions

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Abstract

In this paper, I provide an alternate formula to use for hyperbolic angles utilizing the unit hyperbola and hyperbolic trigonometry. This hyperbolic angle, commonly denoted θ, is used as an input for functions such as $\sinh(\theta)$ and $\cosh(\theta)$, and thus the reformulation will provide alternate means of using these hyperbolic trigonometric functions.

1 Introduction

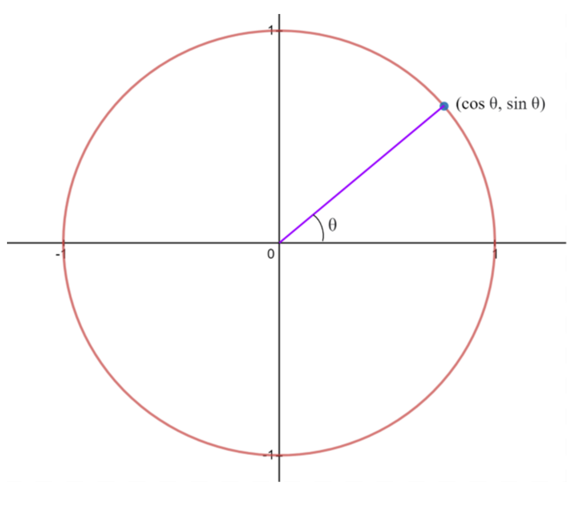
The subject of heavy use in fields such as physics in calculating catenary curves, hyperbolic trigonometric functions serve as an alternative to ordinary trigonometric functions when it comes to dealing with curves.

As functions, hyperbolic trigonometric functions take the input of a hyperbolic angle, and so in this research paper, I seek to calculate the hyperbolic angle. In other words, the following is my overall research question: what is the simplified equation for the hyperbolic angle of hyperbolic trigonometric functions? In the hopes of that strengthening the definition of hyperbolic trigonometric functions, I seek to find such a simplified formula.

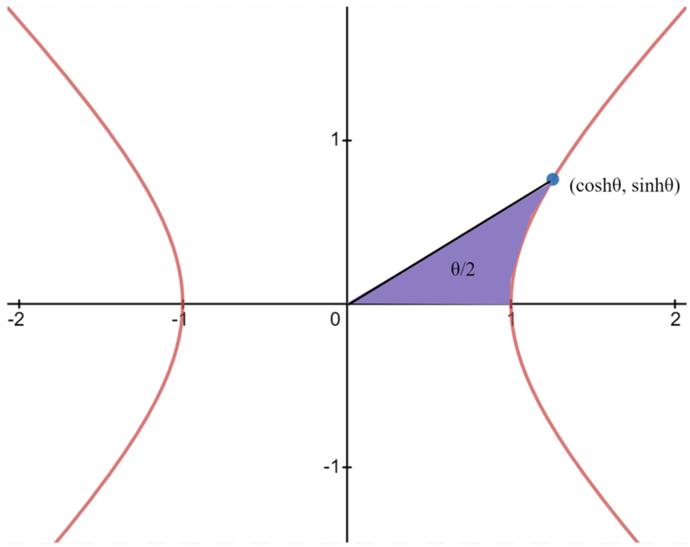
2 The Definition of Hyperbolic Trigonometry

Before any analysis of hyperbolic trigonometric functions is done, they ought to first be defined. Most of the hyperbolic trigonometric functions (tanh, coth, sech, csch) are derived from two primary functions: sinh and cosh. These two functions are quite similar in definition to the normal trigonometric functions of sine and cosine. There are many ways to define all of these functions, but for now, they shall be defined graphically.

Sine and cosine are both geometrically defined around the unit circle, which is a circle of radius one centered around the origin. The values of cosine and sine are then the x and y values of points on the unit circle, respectively. In other words, the circle is represented by the equation $x^2 + y^2 = 1$, with any point on the circle being $(\cos(\theta), \sin(\theta))$, as demonstrated in Diagram 1.



Hyperbolic trigonometric functions, on the other hand, are defined instead along the hyperbola (the "unit hyperbola"), with the newly defined cosh and sinh being represented by x and y, respectively, 3 meaning each point on the unit hyperbola can be represented by $(\cosh(\theta), \sinh(\theta))$. In this case, the graphical representation of the constant θ is not as simple as the angle formed by the line stretching from the origin to the point rather it is double the area created by the purple region shown in Diagram 2. For the sake of gaining a deeper mathematical understanding, θ will now be derived from Diagram 2.



3 Background Proofs

Various useful lemmas/theorems will now be derived to make the integration process more efficient.

Lemma 3.1. $\tan(\sec^{-1}(x)) = \sqrt{x^2 - 1}$

Proof. It is well-established that $\tan^2(x) + 1 = \sec^2(x)$. Therefore:

$$\tan^2(x) + 1 = \sec^2(x) \Rightarrow \tan^2(x) = \sec^2(x) - 1 \Rightarrow \tan(x) = \sqrt{\sec^2(x) - 1}$$

Doing a change of variables with $y = \sec^2(x)$, we have:

$$\tan(y) = \sqrt{\sec^2(y) - 1} = \sqrt{(\sec(\sec^{-1}(x))^2 - 1)} = \sqrt{x^2 - 1}$$

□

Lemma 3.2. $\cosh(x) = \sqrt{\sinh^2(x) + 1}$

Proof. From McMahon (2015), we know that $\cosh^2(x) - \sinh^2(x) = 1$, and so:

$$\cosh^2(x) - \sinh^2(x) = 1 \Rightarrow \cosh(x) = \sqrt{\sinh^2(x) + 1}$$

□

Lemma 3.3 (Integration by Parts)**.** $\int udv = vu - \int vdu$

Proof.

$$\int \frac{d}{dx}[f(x)g(x)]dx = \int f'(x)g(x)dx + \int f(x)g'(x)dx$$

Then, utilizing the Fundamental Theorem of Calculus:

$$\Rightarrow f(x)g(x) = \int f'(x)g(x)dx + \int f(x)g'(x)dx$$

$$\Rightarrow \int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

Substituting $f(x) = u$ and $g(x) = v$, we yield the desired result. $\square$

4 Finding the Value of Theta

Firstly, the line connecting the origin to the point $(\cosh(\theta), \sinh(\theta))$ can be written as $y = \left(\frac{\sinh(\theta)}{\cosh(\theta)}\right) x$. Next, in order to find the area, we can integrate.

Remark. As we integrate, θ will be treated as a constant.

We, therefore, can graphically derive the equation we seek to solve as follows:

$$\frac{\theta}{2} = \int_0^1 \left(\frac{\sinh(\theta)}{\cosh(\theta)}\right) x dx + \int_1^{\cosh(\theta)} \left(\left(\frac{\sinh(\theta)}{\cosh(\theta)}\right) x - f(x)\right) dx \tag{1}$$

where $f(x)$ represents the part of the equation of $x^2 - y^2 = 1$ that lies in the first quadrant. We must, therefore, first find $f(x)$:

$$x^2 - y^2 = 1 \Rightarrow y = \pm\sqrt{x^2 - 1} \text{ where } x \in (-\infty, -1] \cup [1, \infty)$$

$$\Rightarrow f(x) = \sqrt{x^2 - 1} \text{ where } x \in [1, \infty)$$

Therefore, plugging this in, we can rewrite our initial formula:

$$\frac{\theta}{2} = \int_0^1 \left(\frac{\sinh(\theta)}{\cosh(\theta)}\right) x dx + \int_1^{\cosh(\theta)} \left(\left(\frac{\sinh(\theta)}{\cosh(\theta)}\right) x - \sqrt{x^2 - 1}\right) dx \tag{2}$$

Grouping the integrals and simplifying, we get:

$$\frac{\theta}{2} = \left(\frac{\sinh(\theta)}{\cosh(\theta)}\right) \int_0^{\cosh(\theta)} x dx - \int_1^{\cosh(\theta)} \sqrt{x^2 - 1} dx \tag{3}$$

The first integral simplifies quickly:

$$\frac{\theta}{2} = \frac{\sinh(\theta)\cosh(\theta)}{2} - \int_1^{\cosh(\theta)} \sqrt{x^2 - 1} dx \tag{4}$$

The second integral is not quite as simple, and will be given its own sequence of subsections.

4.1 Integration of $\int_1^{\cosh(\theta)} \sqrt{x^2 - 1} dx$

First, we can make use of change of variables, substituting $x = \sec(w)$:

$$\int_1^{\cosh(\theta)} \sqrt{x^2 - 1} dx = \int_0^{\sec^{-1}(\cosh(\theta))} \sec(w)\tan(w)\sqrt{\sec^2(w) - 1} dw \tag{5}$$

From here on, let $a = 0$ and $b = \sec^{-1}(\cosh(\theta))$. We can then substitute in $\tan^2(w) = \sec^2(w) - 1$:

$$\int_a^b \sec(w)\tan(w)\sqrt{\sec^2(w) - 1} dw = \int_a^b \sec(w)\tan(w)\sqrt{\tan^2(w)} dw \tag{6}$$

This can then be simplified further by once again applying the same trigonometric identity:

$$\int_a^b \tan^2(w)\sec(w) dw = \int_a^b \sec^3(w) dw - \int_a^b \sec(w) dw \tag{7}$$

Each integral can then be solved independently.

Integration of $\int_a^b \sec(w)dw$ and $\int_a^b \sec^3(w)dw$

Lemma 4.1. $\int_a^b \sec(w)dw = ln|\sec(w)+\tan(w)|\Big|_a^b$

Proof. Firstly:

$$\int_a^b \sec(w)dw = \int_a^b \sec(w)\left(\frac{\sec(w)+\tan(w)}{\sec(w)+\tan(w)}\right)dw = \int_a^b \frac{\sec^2(w)+\sec(w)\tan(w)}{\sec(w)+\tan(w)}dw$$

Next, using a change of variable of $u = \sec(w)+\tan(w)$, and letting a_u and b_u denote the new upper and lower bounds under this substitution, we get:

$$= \int_{a_u}^{b_u}\left(\frac{\sec^2(w0+\sec(w)\tan(w))}{u}\right)\left(\frac{du}{\sec^2(w)+\sec(w)\tan(w)}\right) = \int_{a_u}^{b_u}\frac{1}{u}du = \ln|u|\Big|_{a_u}^{b_u}$$

$$= \ln|\sec(w)+\tan(w)|\Big|_a^b$$

□

Lemma 4.2. $\int_a^b \sec^3(w)dw = \frac{1}{2}\tan(w)\sec(w)\Big|_a^b + \frac{1}{2}\ln|\sec(w)+\tan(w)|\Big|_a^b$

Proof. First, we can apply Lemma 3.3 with $dv = \sec^2(w)$ and $u = \sec(w)$:

$$\int_a^b \sec^3(w)dw = \tan(w)\sec(w)\Big|_a^b - \int_a^b \tan^2(w)\sec(w)dw$$

$$= \tan(w)\sec(w)\Big|_a^b - \int_a^b (\sec^2(w)-1)\sec(w)dw$$

$$= \tan(w)\sec(w)\Big|_a^b - \int_a^b \sec^3(w)dw + \int_a^b \sec(w)dw = \int_a^b \sec^3(w)dw$$

$$\Rightarrow \int_a^b \sec^3(w)dw = \frac{1}{\tan}(w)\sec(w)\Big|_a^b + \frac{1}{2\int_a^b \sec(w)dw}$$

Using Lemma 4.1, we achieve the desired result. □

4.2 Tying Sums Together

Substituting results from lemmas 4.1 and 4.2, as well as substituting back in the actual bounds, we net:

$$\int_1^{\cosh(\theta)} \sqrt{x^2-1}dx = \left[\frac{1}{2}\tan(w)\sec(w) - \frac{1}{2}\ln|\sec(w)+\tan(w)|\right]\Big|_0^{\sec^{-1}(\cosh(\theta))} \tag{8}$$

Now, we can simplify this expression by treating each part separately.

4.3 Evaluating $\frac{1}{2}\tan(w)\sec(w) - \frac{1}{2}\ln|\sec(w)+\tan(w)|$ at $w = \sec^{-1}(\cosh(\theta))$ and $w = 0$

Lemma 4.3. $\frac{1}{2}\tan(w)\sec(w) - \frac{1}{2}\ln|\sec(w)+\tan(w)|$ *at* $w = \sec^{-1}(\cosh(\theta))$

$$= \tfrac{1}{2}\sinh(\theta)\cosh(\theta) - \tfrac{1}{2}\ln|\cosh(\theta)+\sinh(\theta)|$$

Proof. We can first plug this in and then simplify:

$$\frac{1}{2}\tan(\sec^{-1}(\cosh(\theta)))\sec(\sec^{-1}(\cosh(\theta))) - \frac{1}{2}ln|\sec(\sec^{1-1}(\cosh(\theta))) + \tan(\sec^{-1}(\cosh(\theta)))|$$

$$= \frac{1}{2}\tan(\sec^{-1}(\cosh(\theta)))\cosh(\theta) - \frac{1}{2}\ln|\cosh(\theta) + \tan(\sec^{-1}(\cosh(\theta)))|$$

Using Lemma 3.1, we get:

$$= \frac{1}{2}\cosh(\theta)\sqrt{\cosh^2(\theta) - 1} - \frac{1}{2}\ln\left|\cosh(\theta) + \sqrt{\cosh^2(\theta) - 1}\right|$$

Consequently, Lemma 3.2 nets us:

$$= \frac{1}{2}\sinh(\theta)\cosh(\theta) - \frac{1}{2}\ln|\cosh(\theta) + \sinh(\theta)|$$

which completes the proof. □

Lemma 4.4. $\frac{1}{2}\tan(w)\sec(w) - \frac{1}{2}\ln|\sec(w) + \tan(w)|$ *at* $w = 0$

$$= 0$$

Proof. Plugging in, we simplify as follows:

$$\frac{1}{2}\tan(0)\sec(0) - \frac{1}{2}\ln|\sec(0) + \tan(0)| = \frac{1}{2}\cdot 0 \cdot 1 - \frac{1}{2}\ln|1 + 0| = 0$$

□

Therefore, we have the following equation:

$$\int_1^{\cosh(\theta)} \sqrt{x^2 - 1}dx = \frac{1}{2}\sinh(\theta)\cosh(\theta) - \frac{1}{2}\ln|\cosh(\theta) + \sinh(\theta)| \tag{9}$$

4.4 All-Encompassing Area

Putting both parts together, we now have:

$$\theta = \ln|\cosh(\theta) + \sinh(\theta)| \tag{10}$$

Quite an elegant solution.

5 Conclusion

To take the final result further, recall that, along the unit hyperbola, $x = \cosh(\theta)$ and $y = \sinh(\theta)$. Recall also that $y = f(x) = \sqrt{x^2 - 1} \ni x \in [1, \infty)$. Therefore, we get:

$$\theta = \ln\left|x + \sqrt{x^2 - 1}\right| : x \in [1, \infty) \tag{11}$$

One can think of this as an equation for $\cosh^{-1}(x)$ defined on $x \in [1, \infty)$.

6 Acknowledgments

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References

[1] James McMahon (2015). *Hyperbolic Functions*, Creative Media Partners, LLC.